

ON INFINITE GROUPS ADMITTING A FAITHFUL IRREDUCIBLE REPRESENTATION

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ABSTRACT. Necessary and sufficient conditions for a group to possess a faithful irreducible representation are investigated. We consider locally finite groups and groups whose socle is essential.

1. INTRODUCTION

In 1911, Burnside [B, Note F] posed the problem of finding necessary and sufficient conditions for a finite group to admit a faithful irreducible complex representation. Several solutions have appeared since then. See [Sz] for a detailed history of the subject.

The general case of Burnside's problem, when the group and the ground field are arbitrary, is considerably harder. Partial solutions appear in [T] and [Sz]. It should be noted that [T, Theorem 1] as well as [Sz, Theorem 1.2] require finiteness conditions. In this paper we obtain the following general criterion, where all finiteness conditions have been removed.

We fix throughout an arbitrary field k . All modules will be assumed to be from the left.

Given a group G , let $\text{soc}(G)$ stand for the subgroup of G generated by all minimal normal subgroups of G , and $A(G)$ for the subgroup of G generated by all minimal normal subgroups of G that are torsion abelian. A normal subgroup N of G is said to be essential if every nontrivial normal subgroup of G intersects N nontrivially. Recall that G is locally cyclic (resp. finite) if every finitely generated subgroup of G is cyclic (resp. finite), and that $\text{core}_G(C)$ stands for the intersection of all normal subgroups of G contained in a given subgroup C of G . We write $\Pi(G)$ for the set of all primes p such that $a^p = 1$ for some $a \in G$, $a \neq 1$.

Theorem 1.1. *Let G be an arbitrary group such that $\text{soc}(G)$ is essential and $\text{char}(k) \notin \Pi(\text{soc}(G))$. Then G has a faithful irreducible representation over k if and only if there is a subgroup C of $A(G)$ such that $A(G)/C$ is locally cyclic and $\text{core}_G(C) = 1$.*

For locally finite groups we have the following stronger version, where $\Pi'(G)$ stands set of all primes p such that G has a nontrivial normal p -subgroup.

Theorem 1.2. *Let G be a locally finite group where $\text{soc}(G)$ is essential. Then G has a faithful irreducible representation over k if and only if $\text{char}(k) \notin \Pi'(G)$ and there is a subgroup C of $A(G)$ such that $A(G)/C$ is locally cyclic and $\text{core}_G(C) = 1$.*

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For sufficiency in both cases we rely on [Sz, Theorem 1.1]. We are more concerned here with the necessity of the stated conditions. In this regard, our main tool to prove Theorem 1.1 is the following result, stated under no finiteness conditions.

Theorem 1.3. *Let G be a group with a torsion abelian subgroup A . Suppose that $\text{char}(k) \notin \Pi(A)$ and G admits a faithful irreducible module V over k . Then there is a subgroup C of A such that A/C is locally cyclic and $\text{core}_G(C) = 1$.*

The following additional tool is required to prove Theorem 1.2.

Theorem 1.4. *Suppose that k has prime characteristic p and let G be a locally finite group with a normal p -subgroup P . Then P acts trivially on every irreducible kG -module V .*

Theorems 1.3 and 1.4 also allow us to obtain results on irreducible representations of soluble torsion groups.

Theorem 1.5. *Let G be a torsion soluble group.*

(a) *If G has a faithful irreducible representation over k , then for any abelian normal subgroup A of G we have $\text{char}(k) \notin \Pi(A)$ and there is a subgroup C of A such that A/C is locally cyclic and $\text{core}_G(C) = 1$.*

(b) *If G has a maximal abelian normal subgroup A such that $\text{char}(k) \notin \Pi(A)$ and there is a subgroup C of A such that A/C is locally cyclic and $\text{core}_G(C) = 1$, then G has a faithful irreducible representation over k .*

Evidently, the necessary and sufficient conditions obtained in the above theorem may be amalgamated in the following criterion.

Corollary 1.6. *Let G be a torsion soluble group.*

(a) *If G is abelian, then G has a faithful irreducible representation over k if and only if $\text{char}(k) \notin \Pi(G)$ and G is locally cyclic.*

(b) *If G is nonabelian, then G has a faithful irreducible representation over k if and only if G has a maximal abelian normal subgroup A such that $\text{char}(k) \notin \Pi(A)$ and there is a subgroup C of A such that A/C is locally cyclic and $\text{core}_G(C) = 1$.*

The discussion is illustrated by several examples. In particular, if k has prime characteristic p and $G = F_p(t)^+ \rtimes F_p(t)^*$, then G is shown to have a faithful irreducible module over k . As indicated in §3, this feature of $F_p(t)^+ \rtimes F_p(t)^*$ serves as counterexample to numerous results when their respective finiteness conditions are removed. This applies, in particular, to Theorem 1.2, where local finiteness cannot be eliminated, to Theorem 1.4, whose hypothesis cannot be relaxed to ask that P , but not necessarily G , be locally finite, and to the following necessary criterion for a group to have a faithful irreducible representation.

Theorem 1.7. *Let G be a group having a faithful irreducible module V over k . Let A be a normal, torsion, abelian subgroup of G . Suppose that*

$$(1.1) \quad [G : C_G(A)] < \infty.$$

Then $\text{char}(k) \notin \Pi(A)$.

The finiteness condition (1.1) is imposed to ensure that V has an irreducible $kC_G(A)$ -submodule, which implies the existence an irreducible kA -submodule of V . In general, no such submodules exist.

2. PROOFS

To prove Theorem 1.4 we will rely on the following well-known special case.

Theorem 2.1. *Suppose k has prime characteristic p and let G be a group with a finite, normal, p -subgroup P . Then P acts trivially on every irreducible kG -module V .*

Proof of Theorem 1.4. Let $a \in P$. We need to show that $1 - a \in J(kG)$. This means that $1 - \beta(1 - a)$ is left invertible in kG for every $\beta \in kG$. Given such β , let H be the subgroup of G generated by a and the support of β . Since G is locally finite and H is finitely generated, H is a finite subgroup of G . But $a \in P \cap H$, which is a normal p -subgroup of the finite group H . By Theorem 2.1, $P \cap H$ acts trivially on every irreducible kH -module, so $1 - a \in J(kH)$. Thus $1 - \beta(1 - a)$ is left invertible in kH and hence in kG . \square

The proof of Theorem 1.7 will require the following obvious result.

Lemma 2.2. *Let $N \trianglelefteq G$ be groups and let V be an irreducible kG -module. Suppose N has a nonzero fixed point in V . Then N acts trivially on V .*

Given an abelian group A and a prime p , we let A_p stand for the subgroup of A consisting of all $a \in A$ such that $a^{p^m} = 1$ for some $m \geq 0$.

Proof of Theorem 1.7. Since $[G : C_G(A)] < \infty$, [Pa, Theorem 7.2.16] ensures the existence of an irreducible $kC_G(A)$ -submodule, say U , of V . Since U is irreducible, $D = \text{End}_{kC_G(A)}(U)$ is a division ring.

Suppose k has prime characteristic p . Given $a \in A_p$, the element $1 - a \in kA_p$ is nilpotent, so its image in D must be 0. It follows that A_p acts trivially on U . As A_p is a characteristic subgroup of A , we infer from Lemma 2.2 that A_p acts trivially on V . Since V is a faithful G -module, we deduce that A_p is trivial. \square

We will make use of the following two results in order to prove Theorem 1.3.

Lemma 2.3. *Let A be a torsion abelian group. Let J be a maximal ideal of kA and set $J^\dagger = (J + 1) \cap A$. Then the quotient group A/J^\dagger is locally cyclic.*

Proof. As J is maximal, $F = kA/J$ is a field and we have a natural homomorphism $A \rightarrow F^*$ with kernel J^\dagger . As the torsion subgroup of F^* is locally cyclic, the assertion follows. \square

Proposition 2.4. *Let A be a torsion abelian group. Suppose that $\text{char}(k) \notin \Pi(A)$. Then for any kA -module $V \neq 0$ there is a maximal ideal J of kA such that $JV \neq V$.*

Proof. Given $0 \neq a \in V$, Zorn's Lemma ensures the existence of a submodule U of V that is maximal with respect to $a \notin U$. It is sufficient to show that $J = \text{Ann}_{kA}(V/U)$ is a maximal ideal of kA .

Suppose J is not maximal and hence kA/J is not a field. This easily implies that there is a finite subgroup X of A such that kX/J_X is not a field, where $J_X = kX \cap J = \text{Ann}_{kX}(V/U)$, and hence $J_X = \text{Ann}_{kX}(V/U)$ is not a maximal ideal of kX .

Since $\text{char}(k) \notin \Pi(A)$, we have $1 = e_1 + \cdots + e_m$, where e_1, \dots, e_m are orthogonal idempotents in kX and each $S_i = e_i kX$ is a simple ideal of kX . We then have $V/U = e_1 \cdot (V/U) \oplus \cdots \oplus e_m \cdot (V/U)$, where the annihilator of each nonzero component $e_i \cdot (V/U)$ is a maximal ideal of kX , namely the sum of all S_j with $j \neq i$. As

$\text{Ann}_{kX}(V/U)$ is not a maximal ideal of kX , there must be at least two nonzero components. On the other hand, since $a \notin U$, we see that $a + U$ can belong to at most one component. Thus there is at least one i such that $e_i(V/U) \neq 0$ and $a + U \notin e_i(V/U)$. As kA is commutative, we readily see that $e_i V + U$ is a kA -submodule of V . As such, it properly contains U and $a \notin e_i V + U$, against the maximality of U . Thus, a contradiction is obtained and hence $J = \text{Ann}_{kA}(V/U)$ is a maximal ideal of kA . \square

Example 3.4 shows that the condition $\text{char}(k) \notin \Pi(A)$ cannot be removed from Proposition 2.4.

Proof of Theorem 1.3. By Proposition 2.4, there is a maximal ideal J of kA such that $JV \neq V$. Set $C = J^\dagger$, as in Lemma 2.3, which ensures that the quotient group A/C is locally cyclic.

Let $N = \text{core}_G(C)$. It follows from the definition of J^\dagger that $(1 - J^\dagger) \subseteq J$ and hence, as $N \leq C = J^\dagger$, we have $(1 - N) \subseteq J$. Therefore, as $JV \neq V$, we have $(1 - N)V \neq V$. Since N is a normal subgroup of G , we can show that $(1 - N)V$ is a kG -submodule of kG . Then, as the module V is irreducible and $(1 - N)V \neq V$, we have $(1 - N)V = 0$, which means that N acts trivially on V . As V is faithful, we conclude that $N = 1$. \square

We have not been able to determine whether the condition $\text{char}(k) \notin \Pi(A)$ can be removed from Theorem 1.3, specially when A is a normal subgroup of G and $[G : A]$ is not necessarily finite.

Proof of Theorem 1.5. (a) It is well-known that a torsion soluble group must be locally finite. It therefore follows from Theorem 1.4 that $\text{char}(k) \notin \Pi(A)$. We may now apply Theorem 1.3 to derive the existence of the required subgroup C .

(b) As A is a maximal abelian normal subgroup of a soluble group G , it is easy to see that A is essential in G . By [T, Lemma 2], A has an irreducible representation φ over k such that $\text{Ker } \varphi = C$. Since $C = \text{Ker } \varphi$ contains no nontrivial normal subgroups of G , we see that $A \cap X$ is not contained in $\text{Ker } \varphi$ for any nontrivial normal subgroup X of G , so the assertion follows from [T, Lemma 3].

We are finally in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Suppose first that G has a faithful irreducible G -module over k . Then, by Theorem 1.3, there is a subgroup C of $A(G)$ such that $A(G)/C$ is locally cyclic and $\text{core}_G(C) = 1$. Suppose, conversely, that this condition holds. Then G has a faithful irreducible G -module over k by [Sz, Theorem 1.1]. \square

Proof of Theorem 1.2. Necessity follows from Theorems 1.1 and 1.4, while sufficiency is consequence of [Sz, Theorem 1.1]. \square

3. EXAMPLES

The following three results are well-known.

Theorem 3.1. *Let $N \leq G$ be groups. Then*

- (a) $J(kG) \cap kN \subseteq J(kN)$.
- (b) *If $N \trianglelefteq G$ and N or $[G : N]$ are finite, then*

$$J(kN) = J(kG) \cap kN.$$

- (c) (Clifford) *If $N \trianglelefteq G$ and N or $[G : N]$ are finite, then every irreducible kG -module is a completely reducible kN -module.*

Theorem 3.2. (*Lie-Kolchin*) Let V be a nonzero finite dimensional vector space over k and let A be a subgroup of $\mathrm{GL}(V)$ consisting of unipotent operators. Then A has a nonzero fixed point in V .

Theorem 3.3. (*Nakayama*) Let R be a ring and let V be a finitely generated R -module satisfying $J(R)V = V$. Then $V = 0$.

Example 3.4. Suppose k has prime characteristic p and let $G = C_p \wr C_\infty$ be the wreath product of a cyclic group C_p of order p by an infinite cyclic group C_∞ . Then G has a faithful irreducible module over k .

Proof. It is shown in [Pa, Lemma 9.2.8] that kG has a faithful irreducible module, which is automatically a faithful G -module over k . □

Note 3.5. The Jacobson radical of the group algebra kG of Example 3.4 was studied first in [W].

Example 3.4 illustrates the failure of several results when the stated finiteness conditions are removed. Theorems 1.4, 1.7 and 2.1 above provide examples of this phenomenon. Further examples are given by Theorem 3.1, parts (b) and (c), as well as Theorems 3.2 and 3.3. Indeed, let A be a vector space over F_p with basis $(e_i)_{i \in \mathbb{Z}}$ and let $g \in \mathrm{Aut}(A)$ be given by $g(e_i) = e_{i+1}$ for $i \in \mathbb{Z}$, so that $G = A \rtimes \langle g \rangle$. Let V be the faithful irreducible G -module over k ensured by Example 3.4. If V had an irreducible kA -submodule U then A would act trivially on U by Theorem 1.4, so A would act trivially on V by Lemma 2.2, contradicting the fact that V is faithful. Moreover, it is clear that A acts on V via unipotent operators and, as just indicated, A has no nonzero fixed points in V . Furthermore, we have $J(kG) \cap kA = 0$, by the primitivity of kG , whereas Theorem 1.4 implies that $J(kA)$ is the augmentation ideal of kA . In this regard, since $J(kA)V$ is a kG -submodule of V and A does not act trivially on V , we have $J(kA)V = V$. As $J(kA)$ is the *only* maximal ideal of kA , this shows, in addition, that the condition $\mathrm{char}(k) \notin \Pi(A)$ cannot be removed from Proposition 2.4.

Proposition 3.6. Let G be a group with subgroups H and N satisfying:

- (H1) H has a faithful irreducible representation over k .
- (H2) N is normal and nontrivial, and is included in every nontrivial normal subgroup of G .
- (H3) $N \cap H \neq 1$.

Then G has a faithful irreducible representation over k .

Proof. By (H3), there is a nontrivial $x \in N \cap H$. By (H1), $x - 1 \notin J(kH)$, so $x - 1 \notin J(kG)$ by Theorem 3.1(a). Thus, there is an irreducible kG -module V not acted upon trivially by x . It follows from (H2) that V is a faithful G -module. □

Example 3.7. Suppose k has prime characteristic p . Let D be a division algebra of characteristic p that is not algebraic over its prime field. Then $G = D^+ \rtimes D^*$ has a faithful irreducible module over k .

Proof. Take $N = D^+$ and $H = \langle t^i \mid i \in \mathbb{Z} \rangle \rtimes \langle t \rangle \leq D^+ \rtimes D^*$ in Proposition 3.6, where $t \in D$ is transcendental over F_p , and use Example 3.4. □

Example 3.8. Suppose k has prime characteristic p . Let $M = M(\mathbb{Q}, \leq, F_p)$ be the McLain group, as defined in [M], and set $G = M \rtimes \mathrm{Aut}(M)$. Then G has a faithful irreducible module over k .

Proof. Let A be the subgroup of M generated by $1 + e_{2i, 2i+1}$, $i \in \mathbb{Z}$, let $g \in \text{Aut}(M)$ be induced by the translation $i \mapsto i + 2$ of \mathbb{Q} . Take $H = A \rtimes \langle g \rangle$ and $N = M$ in Proposition 3.6 and use Example 3.4. \square

Since D^+ and M are locally finite p -groups, $D^+ \rtimes D^*$ and $M \rtimes \text{Aut}(M)$ also exhibit the failure of Theorems 2.1, 1.4, 1.7, 3.1, 3.2 and 3.3 when the required finiteness conditions are removed (except that $M \rtimes \text{Aut}(M)$ is unrelated to Theorem 1.7 as M is not abelian). Moreover, Example 3.7 shows that [T, Theorem 1] and [Sz, Theorem 1.2] cease to be valid if their respective finiteness conditions are removed. Indeed, in Example 3.7 the given group G has a minimal normal subgroup A that is an abelian p -group but nevertheless G has a faithful irreducible module over a field of characteristic p (in this regard, note that the socle of $C_p \wr C_\infty$ is trivial).

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